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FINITE ELEMENT ANALYSIS OF NOTCHED TENSILE SPECIMENS IN PLANE STRESS

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PROBLEM STATUS

This is a final report on one phase of the problem; work on the problem continues.

AUTHORIZATION

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FINITE ELEMENT ANALYSIS OF NOTCHED TENSILE SPECIMENS IN PLANE STRESS

INTRODUCTION

Ductile fracture due to the growth and coalescence of voids is prevalent in many structural metals and is governed in large part by the history of mechanical constraint and plastic strain. In particular, McClintock (1) has shown by analysis that there is a strong inverse dependence between fracture strain and transverse stress for hole-growth failure. On the other hand, brittle, slip-initiated cleavage failure propagates across a grain when a critical tensile stress is attained. The generation, evaluation, and application of criteria for such fracture processes depends on a rather precise knowledge of the states of stress and strain during the elastic-plastic deformation preceding failure in the vicinity of notches and cracks. The finite element method has become an effective means of obtaining a complete solution to elastic-plastic problems incorporating realistic constitutive behavior and complicated specimen and loading configurations. Although several two-dimensional, finite element solutions dealing with nonlinear material response have recently appeared in the literature, experimental assessment of their validity has been limited. Analytical and experimental aspects of the problem are frequently treated independently, making comparison of the results difficult.

As part of a study for determining ductile-fracture criteria for structural metals, this report presents a finite element computer program dealing with elastic-plastic deformation in notched tensile specimens under plane stress conditions. Experimental results are given in support of the analysis. The analytical procedure is easily modified to handle similar problems involving plane-strain deformation or axisymmetric geometry.

FINITE ELEMENT FORMULATION

The present analysis adopts first-order triangular elements in which the displacements are taken as linear functions of the spatial coordinates. The Prandtl-Reuss flow rule and von Mises yield criterion describe the plastic response of the material. Although the governing matrix equations have previously been presented in Ref. 2 and 3, for completeness a brief outline of their development will be given. Figure 1 shows a typical triangular element with nodal points i, j, k numbered in counterclockwise order and having coordinates x_i, y_i, etc. The incremental displacement components du and dv, in the x and y directions respectively, are assumed to display a linear variation, over the element, given by

$$du = a_1 + a_2x + a_3y \tag{1a}$$

and

$$dv = a_4 + a_5x + a_6y \ . \tag{1b}$$

The elemental strain increments are then given by

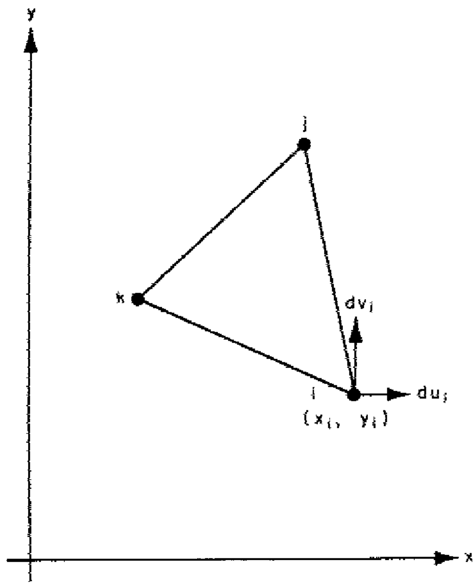

Fig. 1 - Typical triangular element

$$\{d\varepsilon\} \equiv \begin{Bmatrix} d\varepsilon_{xx} \\ d\varepsilon_{yy} \\ d\gamma_{xy} \end{Bmatrix} \equiv \begin{Bmatrix} \dfrac{\partial du}{\partial x} \\ \dfrac{\partial dv}{\partial y} \\ \dfrac{\partial du}{\partial y} + \dfrac{\partial dv}{\partial x} \end{Bmatrix} = \begin{Bmatrix} a_2 \\ a_6 \\ a_3 + a_5 \end{Bmatrix} . \tag{2}$$

Using Eqs. (1), the coefficients a_1, a_2, ..., a_6 may be solved for in terms of the nodal coordinates (x_i, y_i, etc.) and nodal displacement increments (du_i, dv_i, etc.). From Eq. (2) the strain increments can then be expressed by the matrix relationship

$$\{d\varepsilon\} = \frac{1}{2A} [B] \{du\} , \tag{3}$$

where A represents the area of the element, [B] is identified by

$$[B] = \begin{bmatrix} (y_j - y_k) & 0 & (y_k - y_i) & 0 & (y_i - y_i) & 0 \\ 0 & (x_k - x_j) & 0 & (x_i - x_k) & 0 & (x_j - x_i) \\ (x_k - x_j) & (y_j - y_k) & (x_i - x_k) & (y_k - y_i) & (x_j - X_i) & (y_i - y_j) \end{bmatrix} , \tag{4}$$

and {du} is identified by

$$\{du\} = \begin{Bmatrix} du_i \\ dv_i \\ du_j \\ dv_j \\ du_k \\ dv_k \end{Bmatrix} . \tag{5}$$

The general form of the stress-strain relations under plane-stress conditions ($\sigma_{zz} = 0$) is

$$\{d\sigma\} \equiv \begin{Bmatrix} d\sigma_{xx} \\ d\sigma_{yy} \\ d\sigma_{xy} \end{Bmatrix} = [D]\{d\varepsilon\} \ . \tag{6}$$

For linear elastic behavior,

$$[D] = [D_e] = E \begin{bmatrix} 1/1 - \nu^2 & \nu/1 - \nu^2 & 0 \\ & \nu/1 - \nu^2 & 0 \\ (\mathrm{SYM}) & & 1/2(1+\nu) \end{bmatrix} , \tag{7}$$

where E and ν are Young's modulus and Poisson's ratio respectively.

For elastic-plastic deformation, adoption of the Prandtl-Reuss (incremental) stress-strain law in conjunction with the von Mises yield criterion (4) gives

$$[D] = [D_{e-p}] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ & d_{22} & d_{23} \\ (\mathrm{SYM}) & & d_{33} \end{bmatrix} , \tag{8}$$

where the components d_{ij} are expressed in terms of the deviatoric stress components σ'_{ij}, the effective stress $\bar{\sigma} = \sqrt{(3/2)\,\sigma'_{ij}\sigma'_{ij}}$, and the slope H of the effective-stress vs effective-plastic strain curve by

$$d_{11} = \frac{E}{(1-\nu)(1+\nu)\beta}\left[1 + \alpha(\sigma'_{yy})^2 + \frac{2\alpha}{1+\nu}(\sigma_{xy})^2\right] , \tag{9a}$$

$$d_{12} = \frac{E}{(1-\nu)(1+\nu)\beta}\left[\nu - \alpha\sigma'_{xx}\sigma'_{yy} + \frac{2\nu\alpha}{1+\nu}(\sigma_{xy})^2\right] , \tag{9b}$$

$$d_{13} = \frac{-E\alpha\sigma_{xy}}{(1-\nu)(1+\nu)\beta}\left(\frac{\nu\sigma'_{yy} + \sigma'_{xx}}{1+\nu}\right) , \tag{9c}$$

$$d_{22} = \frac{E}{(1-\nu)(1+\nu)\beta}\left[1 + \alpha(\sigma'_{xx})^2 + \frac{2\alpha}{1+\nu}(\sigma_{xy})^2\right] , \tag{9d}$$

$$d_{23} = \frac{-E\alpha\sigma_{xy}}{(1-\nu)(1+\nu)\beta}\left(\frac{\nu\sigma'_{xx} + \sigma'_{yy}}{1+\nu}\right) , \tag{9e}$$

and

$$d_{33} = \frac{E}{(1-\nu)(1+\nu)\beta}\left\{\frac{1-\nu}{2} + \frac{\alpha}{2(1+\nu)}\left[(\sigma'_{xx})^2 + 2\nu\sigma'_{xx}\sigma'_{yy} + (\sigma'_{yy})^2\right]\right\} \tag{9f}$$

with

$$\alpha = \frac{9E}{4(\bar{\sigma})^2 H} , \tag{9g}$$

$$\beta = 1 + \frac{\alpha\left(\sigma'_{yy} + \sigma'_{xx}\right)^2}{1 - \nu^2} + \frac{2\alpha}{1 + \nu}\left[(\sigma_{xy})^2 - \sigma'_{xx}\sigma'_{yy}\right] , \tag{9h}$$

$$\sigma'_{xx} = \frac{2\sigma_{xx} - \sigma_{yy}}{3} , \tag{9i}$$

and

$$\sigma'_{yy} = \frac{2\sigma_{yy} - \sigma_{xx}}{3} . \tag{9j}$$

For purposes of formulating the condition of equilibrium at nodal points throughout the finite element network, the surface tractions on each element are replaced by statically equivalent nodal forces. A relationship is then required between the change in nodal forces and the nodal displacement increments. A linear expression is assumed, having the form

$$\{dF\} = [K]\{du\} , \tag{10}$$

where $\{du\}$ is given by Eq. (5) and $\{dF\}$ is a column matrix containing the x and y components (dU and dV) of the nodal force increments at nodes i, j, and k, i.e.,

$$\{dF\} = \begin{Bmatrix} dU_i \\ dV_i \\ dU_j \\ dV_j \\ dU_k \\ dV_k \end{Bmatrix} . \tag{11}$$

In Eq. (10), $[K]$ is a six-by-six symmetric matrix and is referred to as the elemental stiffness matrix.

The components of the elemental stiffness matrix are determined by imposing an arbitrary nodal displacement increment and equating the work done by the nodal forces to the increase in strain energy of the element. The work done on a typical element during a nodal displacement increment $\{du\}$ in which the nodal forces change by $\{dF\}$ is

$$\delta W = \{F\}^T\{du\} + \frac{1}{2}\{dF\}^T\{du\} , \tag{12}$$

where the superscript T denotes the matrix transpose and $\{F\}$ represents the nodal forces acting on the element initially. Inserting Eq. (10) into Eq. (12) and successively differentiating δW with respect to each component of $\{du\}$ gives

$$\left.\begin{aligned}
\frac{\partial \delta W}{\partial \delta u_i} &= U_i + [k_{11}k_{12}k_{13}k_{14}k_{15}k_{16}]\{du\} \ , \\
\frac{\partial \delta W}{\partial \delta v_i} &= V_i + [k_{21}k_{22}k_{23}k_{24}k_{25}k_{26}]\{du\} \ , \\
&\dots \\
\frac{\partial \delta W}{\partial \delta v_k} &= V_k + [k_{61}k_{62}k_{63}k_{64}k_{65}k_{66}]\{du\} \ ,
\end{aligned}\right\} \tag{13}$$

where the k_{ij} are components of the elemental stiffness $[K]$. The change in strain energy of the element during the displacement increment is

$$\delta E = At \left[\{\sigma\}^T \{d\varepsilon\} + \frac{1}{2} \{d\sigma\}^T \{d\varepsilon\}\right] , \tag{14}$$

where t is the thickness of the element and $\{\sigma\}$ contains the initial stresses. Insertion of Eqs. (3) and (6) into Eq. (14) yields

$$\delta E = \frac{t}{2} \left[\{\sigma\}^T [B]\{du\} + \frac{1}{4A} \{du\}^T [B]^T [D][B]\{du\}\right] . \tag{15}$$

Equating δE and δW, Eq. (15) is inserted into each of Eqs. (13), indicated differentiations are performed, and like multipliers of the incremental displacement components are equated. The result is

$$[K] = \frac{t}{4A} [B]^T [D][B] . \tag{16}$$

In the elastic-plastic range, $[K]$, being a function of $[D_{e-p}]$, depends on the instantaneous stress state in the element as well as on the elastic constants and element geometry.

To establish equilibrium at a typical node i of the finite element network, the applied (external) force increment $\{dR\}$ at the node is equated to the sum of the internal force increments contributed by each of those elements having node i as one of its verticies. The internal force increment $\{dF\}$ at node i is calculated from Eq. (10) for each surrounding element in terms of its elemental stiffness and nodal displacement increments. When equilibrium at each of N nodes is considered in the x and y directions, the system of linear equations

$$\{dR\} = [A]\{du\} \tag{17}$$

is generated. The column matrices $\{dR\}$ and $\{du\}$ are defined by

$$\{du\} = \begin{Bmatrix} du_1 \\ dv_1 \\ \dots \\ du_i \\ dv_i \\ \dots \\ du_N \\ dv_N \end{Bmatrix} \tag{18}$$

and

$$\{dR\} = \left\{ \begin{matrix} dR_{x,1} \\ dR_{y,1} \\ \cdots \\ dR_{x,i} \\ dR_{y,i} \\ \cdots \\ dR_{x,N} \\ dR_{y,N} \end{matrix} \right\}, \tag{19}$$

where $dR_{x,i}$ and $dR_{y,i}$ represent the applied force increments in the x and y directions at node i. The overall stiffness matrix $[A]$ is square (2N by 2N), symmetric, and positive definite.

The nodal displacement increments $\{du\}$ resulting from an applied force system $\{dR\}$ may be obtained by inversion of Eq. (17). The elemental strain and stress increments are then calculated from Eqs. (3) and (6) respectively.

COMPUTER PROGRAM

To implement the solution derived in the previous section, a computer program has been written in FORTRAN 63 to analyze notched specimens loaded in uniaxial tension under plane-stress conditions $(\sigma_{zz} = 0)$.

Following Yamada et al. (5), the elastic-plastic solution is generated by loading the body in incremental steps, each load increment being just sufficient to cause an additional element to become plastic. Initially, an elastic solution is obtained, and that element having the largest effective stress is determined; this element yields first. To determine the load increment required to cause the next element to yield, the following procedure is adopted. A unit load is applied, and the resulting changes in stress components are calculated for each elastic element. For a typical elastic element, the current deviatoric stresses are denoted by σ'_{ij}, and $\Delta\sigma'_{ij}$ represents the change in deviatoric stresses due to the unit load. For a small load increment, the elemental stress and load increments are proportional, and if ΔP is the applied load increment necessary to just cause the element to yield, the associated deviatoric stress increments will be $\Delta P \Delta\sigma'_{ij}$. According to the von Mises yield criterion,

$$(\sigma_y)^2 = \frac{3}{2}\left(\sigma'_{ij} + \Delta P \Delta\sigma'_{ij}\right)\left(\sigma'_{ij} + \Delta P \Delta\sigma'_{ij}\right) , \tag{20}$$

where σ_y is the uniaxial yield stress. Solution of Eq. (20) for ΔP gives

$$\Delta P = \frac{-\sigma'_{ij}\Delta\sigma'_{ij} + \sqrt{(\sigma'_{ij}\Delta\sigma'_{ij})^2 - (\Delta\sigma'_{ij}\Delta\sigma'_{ij})\left[\sigma'_{kl}\sigma'_{kl} - \frac{2}{3}\sigma_y^2\right]}}{\Delta\sigma'_{ij}\Delta\sigma_{ij}} . \tag{21}$$

Equation (21) is evaluated for each elastic element, and that element having the minimum value of ΔP, i.e., ΔP_{min}, will be the next to yield. The deviatoric stress components in each element are then increased by the amount $\Delta P_{min} \Delta \sigma'_{ij}$, and the applied load is increased by ΔP_{min}. This procedure is repeated until the desired number of elements have yielded.

One quadrant of a double-edge-notched specimen loaded in uniaxial tension is shown in Fig. 2. To determine the elemental stresses corresponding to a unit load, the matrix $\{dR\}$ defined by Eq. (17) is formulated as follows. The applied forces acting at nodes along boundary CD shown in Fig. 2 correspond to an applied uniaxial tensile stress of 1000 (unit loading). Applied forces at all interior nodes and those along the free surface boundary AD are zero. The displacement component normal to symmetry axes AB and BC must be zero at all nodes located on these axes. If component I of the incremental displacement matrix $\{du\}$ is specified to be zero, row I and column I of the overall stiffness matrix $[A]$ (except the diagonal component) and component I of $\{dR\}$ are set to zero. This procedure effectively eliminates those equations expressing equilibrium at nodes having specified zero displacements (and unknown force components). Since shear stresses cannot be transmitted across an axis of symmetry, the applied forces at nodes along boundaries AB and BC have a zero component parallel to the boundary.

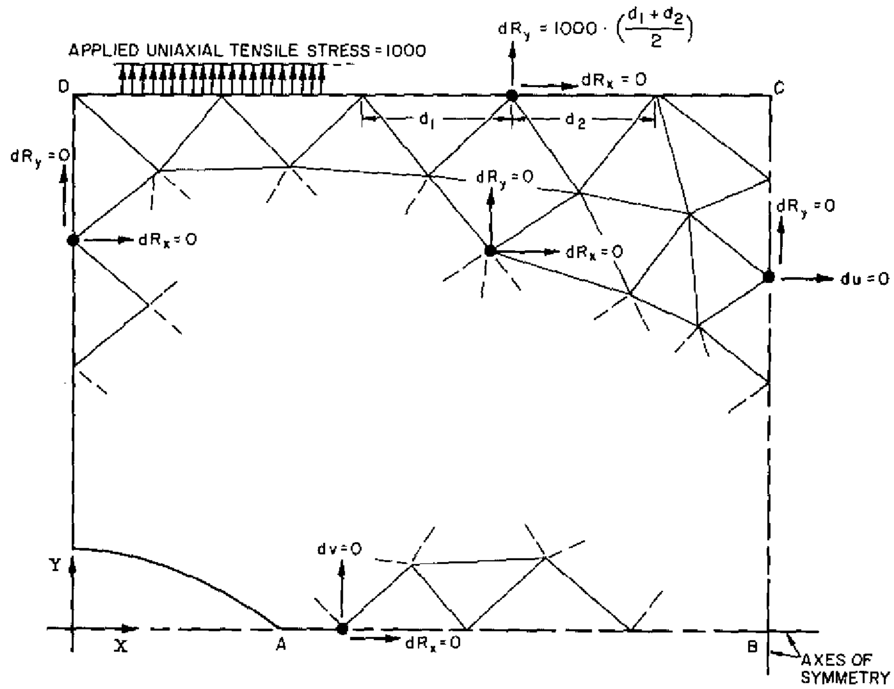

Fig. 2 - Specimen geometry and finite element boundary conditions

Having formed $\{dR\}$, the overall stiffness matrix $[A]$ is constructed from the elemental stiffness matrices by successive application of Eq. (10) to each element of the network. If, for example, element M has nodes i, j, and k, the rows of its elemental stiffness matrix are recorded in the appropriate locations of rows $2i-1$, $2i$, $2j-1$, $2j$, $2k-1$, and $2k$ of matrix $[A]$. These entries represent the contribution of element M to the x and y internal force components acting at nodes i, j, and k respectively. When the internal forces from all elements have been recorded, the matrix $[A]$ is completely formed. After solution of Eq. (17) for displacement increments, the elemental strain and stress increments are readily obtained from Eqs. (3) and (6) respectively.

The effective-stress vs effective-plastic-strain curve of the material is approximated by a series of linear segments, the terminal points of which are entered as input data. The stress-strain path for each plastic element is maintained within predetermined limits of the prescribed flow curve by an iterative scheme for selecting the appropriate tangent modulus for each load cycle (load increment). Figure 3 illustrates the procedure. At the beginning of a load cycle the effective stress and effective plastic strain for a typical plastic element are denoted by $\bar{\sigma}_0$ and $\bar{\varepsilon}_0$ respectively. As a first approximation to the tangent modulus, the value H_0 corresponding to the slope of the prescribed flow curve at $\bar{\varepsilon}_0$ is selected. During loading the element follows a linear path with slope H_0 to the terminal point $(\bar{\sigma}_1, \bar{\varepsilon}_1)$. If $\bar{\sigma}(\bar{\varepsilon}_1)$ denotes the effective stress on the prescribed flow curve corresponding to the strain $\bar{\varepsilon}_1$, the degree of deviation (CONV) from the prescribed curve may be expressed by

$$\mathrm{CONV} = \frac{\bar{\sigma}_1 - \bar{\sigma}(\bar{\varepsilon}_1)}{\bar{\sigma}(\bar{\varepsilon}_1)} . \tag{22}$$

If CONV exceeds a critical predetermined value (CRIT), a second approximation H_1 to the tangent modulus is made:

$$H_1 = \frac{\bar{\sigma}(\bar{\varepsilon}_1) - \bar{\sigma}_0}{\bar{\varepsilon}_1 - \bar{\varepsilon}_0} . \tag{23}$$

This procedure is applied for each load cycle until all plastic elements are within the prescribed limits of the given flow curve or until the maximum allowed number of iterations (NITMAX) has been achieved.

To account for changes in geometry during loading, the coordinates of the nodal points and the thicknesses of the elements are updated by the end of each load cycle.

A general flow chart for the program is shown in Fig. 4. The parameter CYC counts the number of load cycles performed in the elastic-plastic range; when CYC equals CYCMAX, computation ends. Entry of CYCMAX equal to zero results in an elastic solution only. A listing of the program as well as further details concerning its use are given in the appendix.

COMPARISON OF EXPERIMENTAL AND ANALYTICAL RESULTS

To assess the accuracy of the computer solution, an arbitrary notch geometry was selected, and experimental and numerical results were compared. Double-edge-notched tensile specimens having the dimensions shown in Fig. 5 were machined from 1/16-in.-thick, 2024-T3 aluminum sheet. The degree of notch root strain was determined as a function of applied loading by placing a miniature (0.030 in. by 0.030 in.), high-elongation, resistance strain gage at the base of each notch root. For each specimen, two active and two temperature-compensating dummy gages were wired in a bridge circuit such that the output reflected the average strain in both notches. The specimens were loaded in tension at a crosshead speed of 0.002 in. per min, and the applied load and notch strain were simultaneously recorded on an X-Y recorder.

The true-stress vs true-strain curve for 2024-T3 aluminum alloy is shown in Fig. 6 and was determined by placing strain gages on unnotched, sheet tensile specimens.

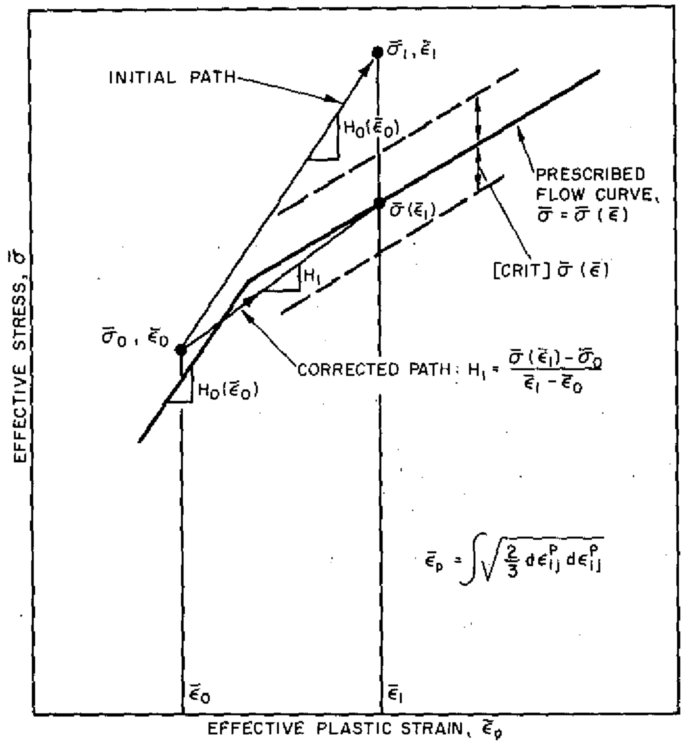

Fig. 3 - Iterative method for selecting the appropriate tangent modulus

Young's modulus, Poisson's ratio, and the proportional limit were determined to be 10.4×10^3 ksi, 0.33, and 45.34 ksi respectively.

From the measured variation of average notch root strain (ε_{nr}) with applied loading, the dependence of the strain concentration factor K_ε and stress concentration factor K_σ on nominal net-section stress (σ_{nom}) was obtained. K_ε and K_σ are defined by

$$K_\varepsilon = \frac{\varepsilon_{nr} E}{\sigma_{nom}} \tag{24a}$$

and

$$K_\sigma = \frac{\sigma_{nr}}{\sigma_{nom}}, \tag{24b}$$

where σ_{nr} is the tangential stress at the notch-root surface. Because the stress state at the notch root surface is simple tension (assuming plane-stress behavior), σ_{nr} was determined from the uniaxial flow curve given in Fig. 6 by reading off the stresses corresponding to the measured values of ε_{nr}.

The finite element grid used to characterize one quadrant of the specimen is shown in Figs. 7 and 8 and contains 395 triangular elements with 227 nodal points. Uniaxial tension is assumed to act along the boundary opposite the minimum section, i.e., at a distance from the notched section equal to 13.5 times the notch root radius. Plane-stress behavior is assumed in all elements. The curve of effective-stress vs effective-plastic strain was approximated by 11 linear segments with terminal points, as shown in Fig. 6.

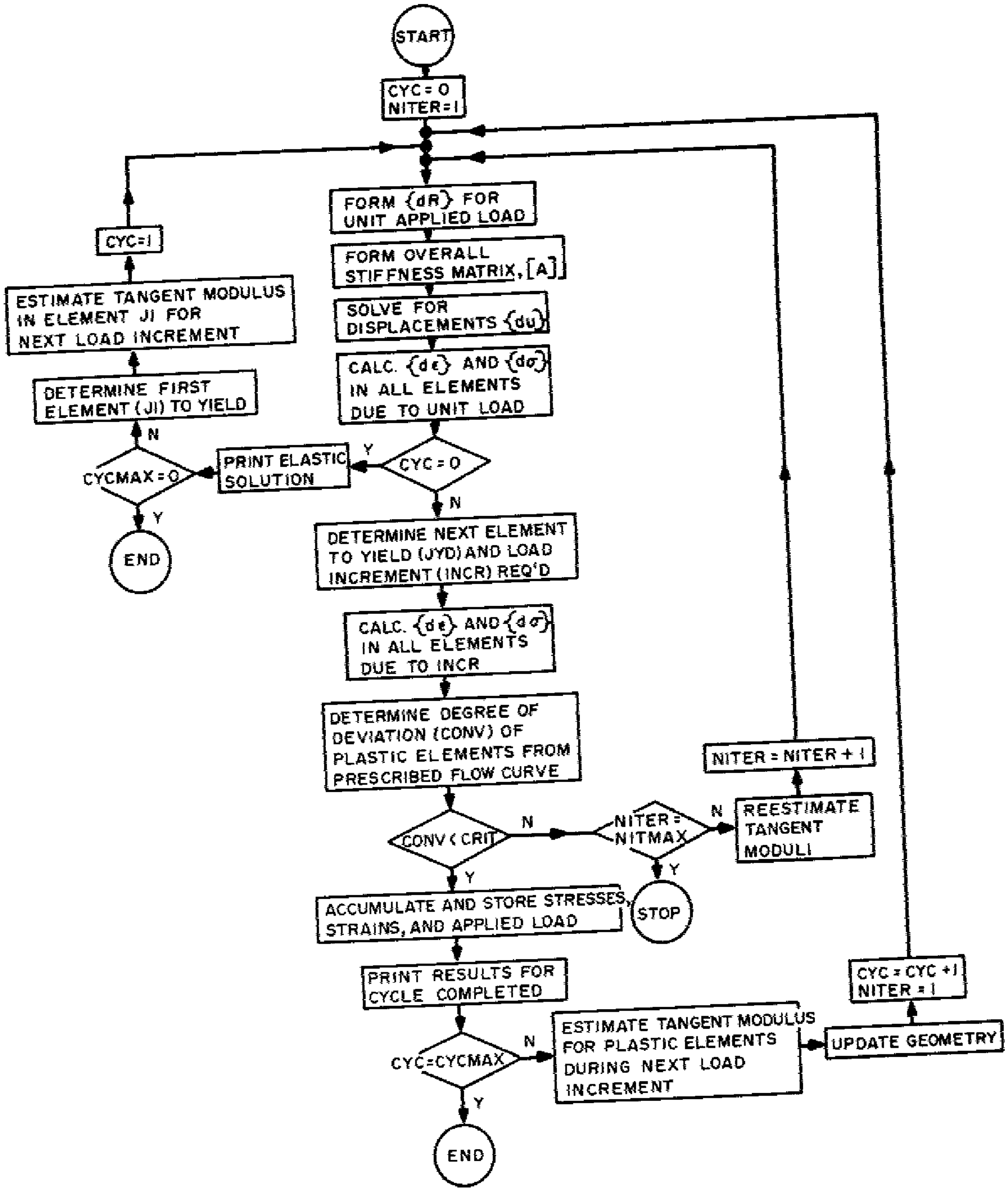

Fig. 4 - Computer program flow chart

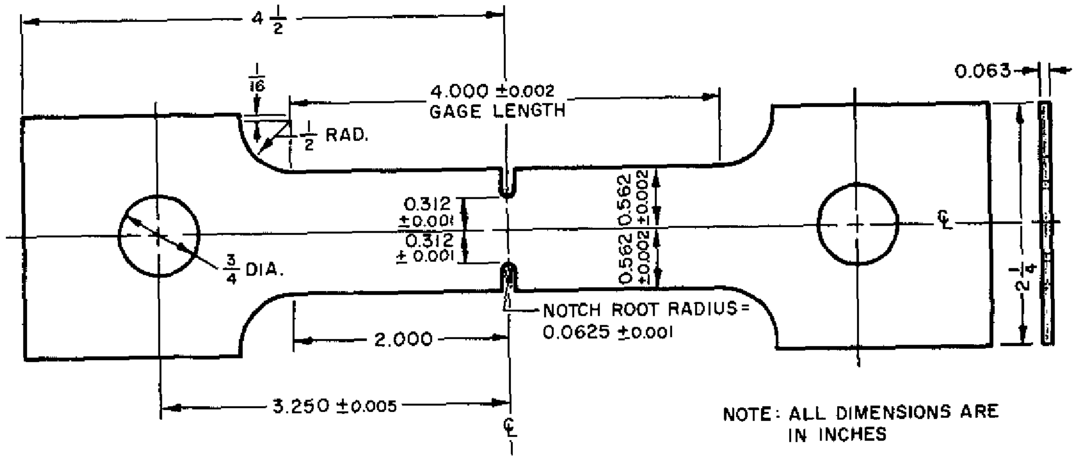

Fig. 5 - Specimen geometry for 2024-T3 aluminum sheet tensile specimen

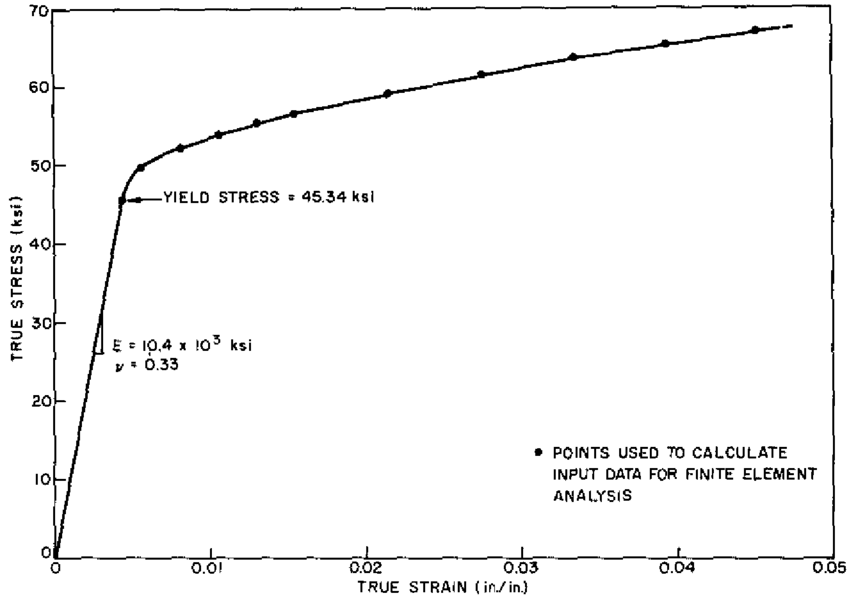

Fig. 6 - True-stress vs true-plastic strain curve for 2024-T3 aluminum

Deviation of the elemental effective stresses from the prescribed flow curve was limited to 0.5 percent throughout the loading history. The solution was carried out beyond general yielding to a value of σ_{nom} equal to 57.17 ksi, at which point 201 elements had become plastic. A complete listing of the input data for this problem is given in the appendix.

DISCUSSION OF RESULT

Throughout the loading history the largest strain in the finite-element mesh was found to be the axial strain ε_y in the small element located at the base of the notch-root surface on the minimum section (Fig. 8). This strain is assumed comparable to that detected experimentally by the strain gages and was used to determine the analytical variation of K_ε with σ_{nom}. The elemental axial stress corresponding to ε_y was used to obtain a similar relationship between K_σ and σ_{nom}.

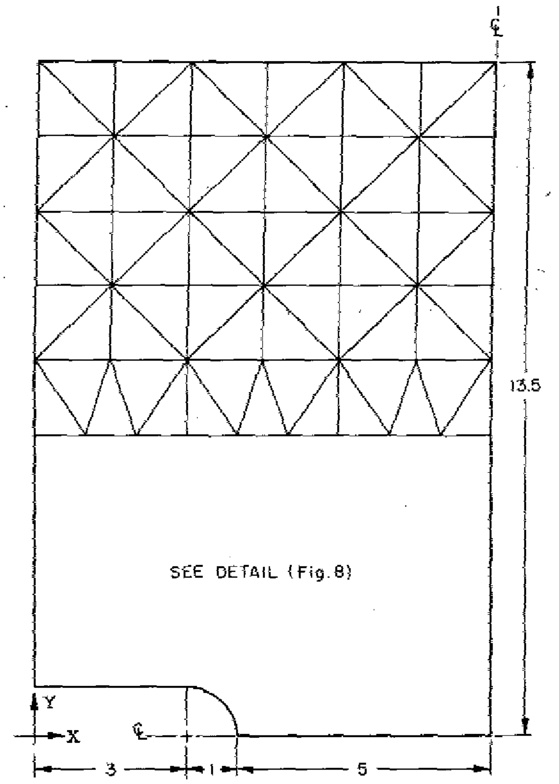

Fig. 7 - Finite element grid for one quadrant of the specimen

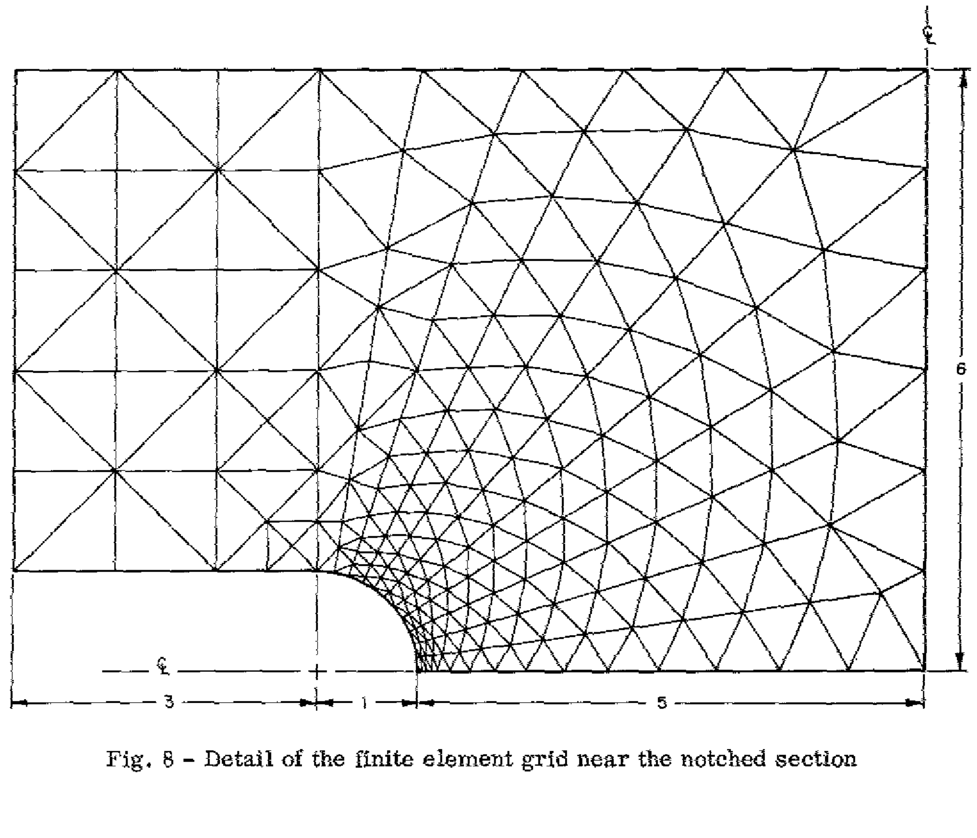

Fig. 8 - Detail of the finite element grid near the notched section

The elastic stress concentration factor K_t determined by finite element analysis is 3.06, which is in good agreement with the experimental values of 2.90 and 3.09. Application of the semiempirical method given in Ref. 6 results in a value of 2.98, which further indicates that the procedures used for deriving the stress and strain concentration factors from the present numerical and experimental results are valid.

The experimental and analytical variations of K_σ and K_ε with σ_{nom} are shown in Fig. 9. As the applied load is increased above that required for initial yielding (σ_{nom} = 14.8 ksi), the strain concentration factor increases while the stress concentration factor decreases, the latter approaching a value of unity. The overall comparison between experimental and numerical results is considered favorable. For any σ_{nom} below general yield (σ_{nom} < 54.5 ksi), the analytical value of K_ε falls within 8 percent of the corresponding mean experimental result. The greatest difference between experimental and theoretical values of K_ε is about 11 percent, which occurs at load levels above general yield. In this loading range, small increases in applied load produce relatively large increases in notch strain, due to the general reduction in stiffness as the plastic zone traverses the minimum section. It is likely that this increased discrepancy reflects an accumulation of error in the analysis because of approximating the loading path by a finite number of linear stress-strain segments. The difference in experimental and analytical values of K_σ beyond general yield is less than 2 percent because of the reduced rate of strain hardening at these high notch-root strains; i.e., a relatively large discrepancy in notch-root strain results in only a minor difference in the corresponding value of notch stress.

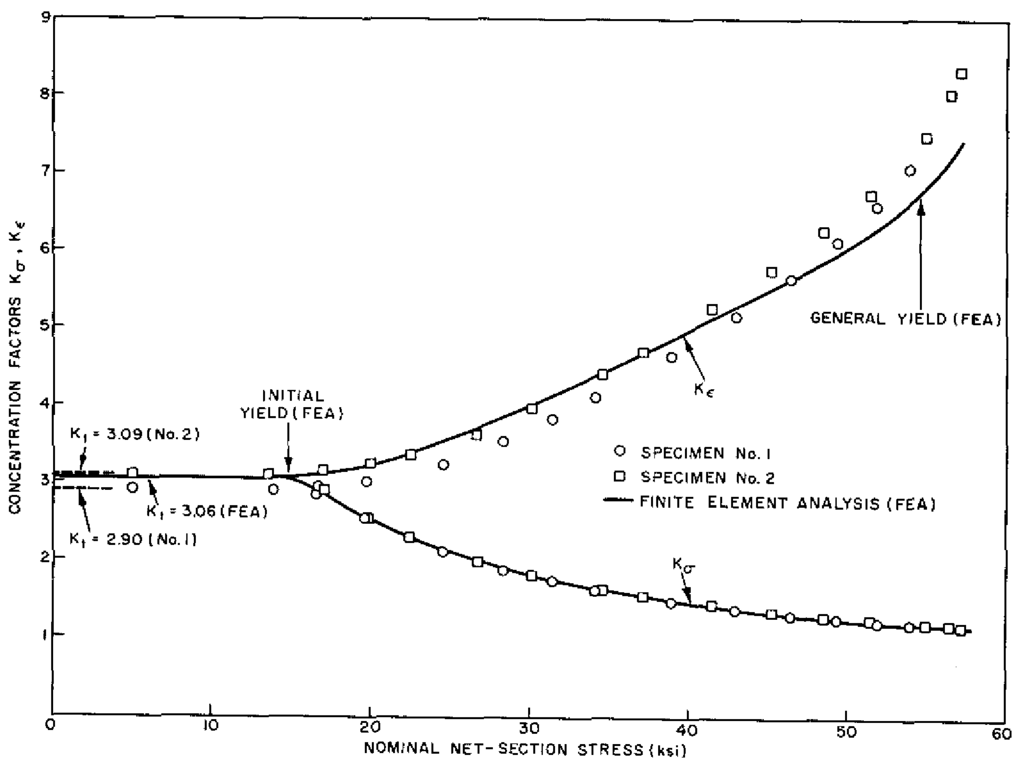

Fig. 9 - Experimental and analytical variation of stress and strain concentration factors with net-section stress

In his analysis of sharply notched bars loaded in pure shear, Neuber (7) found that the elastic stress concentration factor is equal to the geometric mean of the stress and strain concentration factors, i.e., $K_\sigma K_\varepsilon / K_t^2 = 1$. Recently several investigators (8, 9) have suggested that this result may be valid for notched bars loaded in tension under plane-stress conditions. Figure 10 gives the analytical and experimental variation of the quantity $K_\sigma K_\varepsilon / K_t^2$ with σ_{nom}. It is evident that $K_\sigma K_\varepsilon / K_t^2$ is less than unit over most of the loading range investigated, reaching a minimum value of 0.75 to 0.79 at a nominal stress between 29 and 37 ksi. Thus the Neuber relationship is not applicable to plane-stress tensile loading for the notch geometry and stress-strain behavior encountered in the present study.

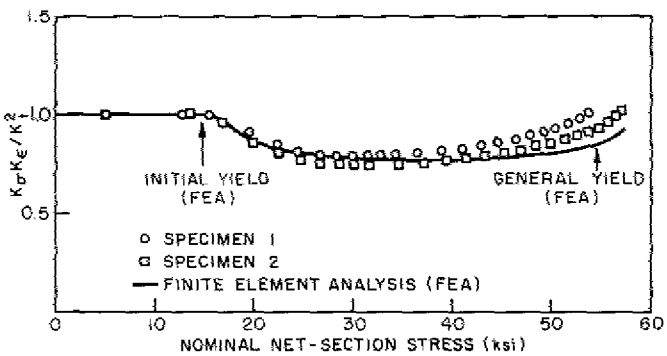

Fig. 10 - Experimental and analytical variation of $K_\sigma K_\varepsilon / K_t^2$ with net-section stress

The finite element analysis predicts that the plastic zone develops in the manner indicated in Fig. 11. Yielding initiates at the notch root and spreads more or less uniformly in all directions for σ_{nom} less than about 40 ksi. At higher loads the plastic zone becomes somewhat kidney shaped, and, as general yielding is approached, growth occurs primarily in that portion most removed from the minimum section. At general yield the plastic zone envelops a central core of elastically strained material.

The axial-stress σ_y and axial-strain ε_y distributions along the minimum section are presented in Fig. 12 for two levels of loading ($\sigma_{nom} = 37.99$ and 51.62 ksi) in the elastic-plastic range. Also shown are the stress and strain distributions at $\sigma_{nom} = 37.99$ ksi if the material were completely elastic. These distributions were obtained from finite element results by assuming that the stress and strain at a given node on the minimum section are given by the average of the values found in surrounding elements. As for the case of purely elastic response, the strain in the elastic-plastic range is maximized at the notch-root surface and decreases uniformly with distance beneath the notch. However, the strain gradients in the plastic zone are considerably greater than those corresponding to purely elastic behavior, which accounts for the increase in K_ε with σ_{nom} shown in Fig. 9.

Examination in Fig. 12 of the stress distribution near the notch reveals that a major effect of elastic-plastic flow (as opposed to linear-elastic response) is a redistribution of load so as to markedly reduce the stress gradients in the plastic zone. Once yielding begins, the axial stress rises slightly with distance below the notch, reaching a maximum value between the notch surface and the elastic-plastic boundary. The extent of this rise depends on σ_{nom}, but it is generally small; σ_y never increases more than 3.5 percent above its notch root value over the load range investigated. The existence of such a

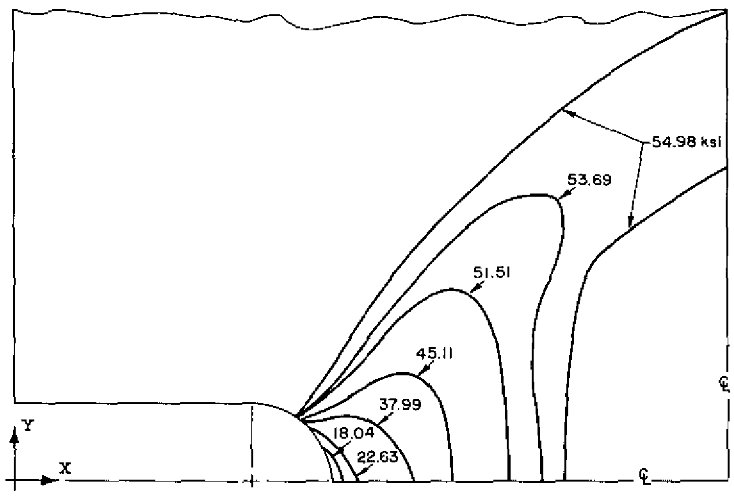

Fig. 11 - Extent of the elastic-plastic boundary at the indicated levels of net-section stress

maximum in axial stress generally depends on the notch geometry, on the degree of strain hardening, and, perhaps, on the yield criterion.

SUMMARY AND CONCLUSIONS

1. A computer program using the finite element method has been written to analyze plane stress and elastic-plastic deformation in notched tensile specimens. Loading is applied incrementally, each increment being of just sufficient magnitude to cause an additional element to become plastic. The Prandtl-Reuss flow rule and von Mises yield criterion are employed to characterize the elastic-plastic response of the material.

2. Numerical results are in reasonably good agreement with experimental notch root strain measurements made on double-edge-notched tensile specimens made from 2024-T3 aluminum (proportional limit = 45.3 ksi) sheet. Analytical and experimental values of the strain concentration factor differ by a maximum of 8 percent at load levels below general yield, i.e., $\sigma_{nom} < 54.5$ ksi.

3. Experimental and numerical results indicate that the Neuber relationship, $K_\sigma K_\epsilon / K_t^2 = 1$, is not applicable to plane-stress tensile loading for the notch geometry and strain-hardening behavior encountered in the present investigation. The value of $K_\sigma K_\epsilon / K_t^2$ was found to be less than unity over the major portion of the load range studied, reaching a minimum value of 0.75 to 0.79.

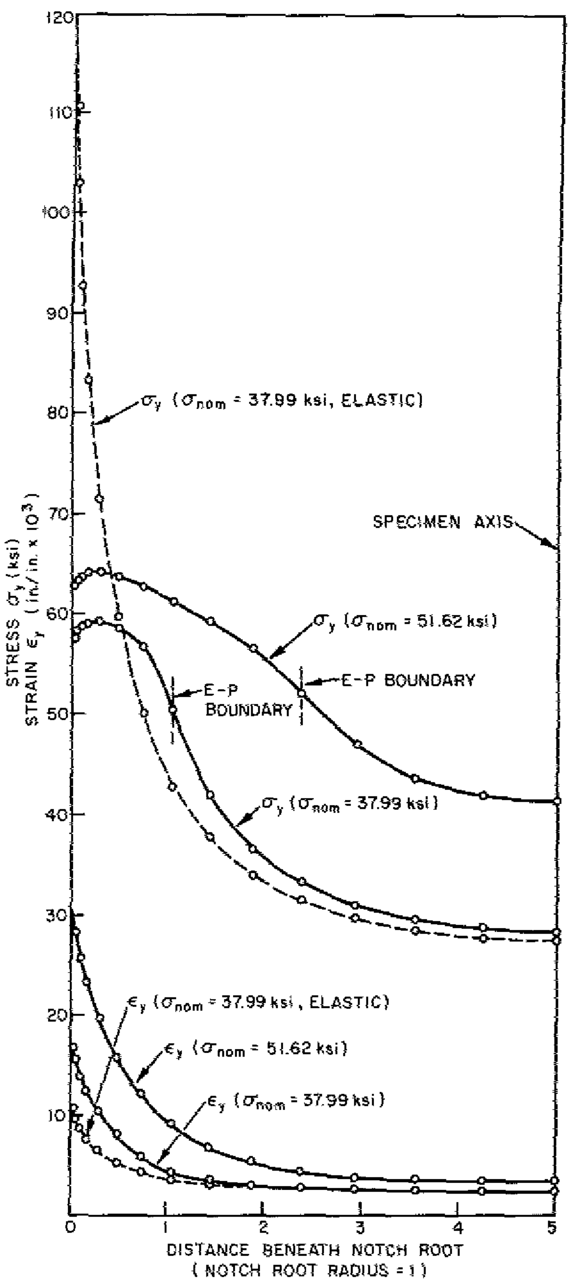

Fig. 12 - Axial stress and strain distribution along the minimum section for net-section stresses of 37.99 and 51.62 ksi

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Appendix

COMPUTER PROGRAM AND INSTRUCTIONS FOR USE

NOTATION

The meaning of the various symbols used in the program and the sizes of arrays are given below.

M	-	Number of elements
N	-	Number of nodes
M1	-	Number of nodes located on the boundary opposite the minimum section (Fig. 2, boundary CD)
M2	-	Number of nodes for which the x displacement is specified to be zero
M22	-	Number of nodes for which the y displacement is specified to be zero
NO	-	Number of linear segments on the specified flow curve
BDW	-	Bandwidth of the overall stiffness matrix. BDW = 4 [maximum difference between adjacent nodal numbers] + 3.
YBAR	-	Uniaxial yield stress
PR	-	Poisson's ratio
E	-	Young's modulus
CRIT	-	Maximum allowable deviation from the prescribed flow curve (see Fig. 3)
CONV	-	Calculated deviation from the prescribed flow curve
CYCMAX	-	Maximum number of load increments to be applied. Entry of CYCMAX equal to zero results in an elastic solution only.
INCR	-	Applied load increment necessary to cause an additional element to yield
CYC	-	Count of the number of cycles (load increments) completed
NITMAX	-	Maximum number of iterations allowed in determining the tangent modulus for any load increment

NITER	- Count of the number of iterations completed for a given load increment
J1	- First element to yield
SBAR	- Effective stress in the element J1 at initial yield
CORD (N, 2)	- CORD (I, 1) and CORD (I, 2) represent the x and y coordinates, respectively, of node I
NP (M, 3)	- NP (I, 1), NP (I, 2), and NP (I, 3) contain the nodal numbers in counterclockwise order for element I
AREA (M)	- AREA (I) contains the current area of element I
GE (3, 3)	- Stress-strain matrix (Eq. (8))
DU (6)	- Incremental nodal displacement matrix (Eq. (5))
DE (3)	- Incremental strain matrix (Eq. (2))
DELM (M, 5)	- DELM (I, 1), DELM (I, 2), DELM (I, 3), DELM (I, 4), and DELM (I, 5) contain ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, and the effective plastic strain, respectively, in element I (accumulated values)
DELM1 (M, 5)	- Strain increments for all M elements resulting from a given load increment
SELM (M, 5)	- SELM (I, 1), SELM (I, 2), SELM (I, 3), SELM (I, 4), and SELM (I, 5) contain σ_{xx}, σ_{yy}, σ_{xy}, σ_{zz}, and the effective stress, respectively, in element I (accumulated values)
SELM1 (M, 5)	- Stress increments for all M elements resulting from a given load increment
B (3, 6)	- Strain-displacement matrix (Eq. (4))
DSPX (M2)	- Nodal numbers of the nodes for which the x displacements are specified to be zero
DSPY (M22)	- Nodal numbers of the nodes for which the y displacements are specified to be zero
DS (3)	- Incremental stress matrix (Eq. (6))
EP (M)	- EP (I) distinguishes elastic and elastic-plastic elements. If EP (I) is zero, element I is elastic; if EP (I) is unity, element I has yielded.
FSPEC (M1, 3)	- FSPEC (I, 2) and FSPEC (I, 3) are the x and y components, respectively, of the applied force acting at node FSPEC (I, 1) which is located on the boundary opposite the notched section. FSPEC (I, 2) and FSPEC (I, 3) correspond to an applied uniaxial tensile stress of 1000 integrated between midpoints of the two segments joining node FSPEC (I, 1)

with its two adjacent boundary nodes. This force system, which is entered as input data, is referred to as a unit load (See Fig. 2).

THKNS (M) - THKNS (I) contains the current thickness of element I

HEFF (M) - HEFF (I) contains the current tangent modulus for element I

$A\left(2N, \frac{BDW+1}{2}\right)$ - This array contains those components of the overall stiffness matrix (Eq. (17)) which lie on and above the main diagonal. Other components need not be recorded, since the overall stiffness matrix is symmetric. The diagonal element in row I of the overall stiffness matrix is contained in A (I, 1).

DELF (2N) - Applied force matrix $\{dR\}$ defined by Eq. (19)

$DELU\left(2N + \frac{BDW-1}{2}\right)$ - The displacement matrix $\{du\}$ defined by Eq. (18). The last (BDW - 1)/2 locations of this array are incorporated for convenience in solving Eq. (17).

H (NO, 3) - H (I, 1) contains the slope of the Ith segment of the prescribed effective-stress vs effective-plastic-strain curve. H (I, 2) and H (I, 3) are the effective stress and effective plastic strain, respectively, at the beginning of the Ith segment.

$\left.\begin{array}{l} B1\left(\frac{BDW+1}{2}\right) \\ C1\left(\frac{BDW+1}{2}, \frac{BDW+1}{2}\right) \end{array}\right\}$ - Arrays used in subroutine SOLVE

SUBROUTINES

Two subroutines, SUBSEM and SOLVE, supplement the main program. For a given value of effective plastic strain (EM1), subroutine SUBSEM returns the corresponding value of effective stress (SEM1) and tangent modulus (HEM1) given by the prescribed flow curve. Using the Gaussian elimination method, subroutine SOLVE* is used to solve the system of linear equations given by Eq. (17). This subroutine takes advantage of the fact that the overall stiffness matrix [A] is symmetric and likely to be highly banded. These characteristics significantly reduce computer storage requirements.

INPUT DATA

The order in which data are read and the formats employed is as follows.

*The author is indebted to Dr. D. J. Krause for providing this subroutine.

Data	Format
PR, BDW, N, M, M2, M22, M1	F4.0, 6(I4)
CYCMAX, NITMAX, CRIT, E, YBAR, NO	2(I4), F5.0, 2(E10.5), I4
First column of array CORD (N,2)	12(F6.4)
Second column of array CORD (N,2)	12(F6.4)
First column of array (NP (M,3)	26(I3)
Second column of arry NP (M,3)	26(I3)
Third column of array NP (M,3)	26(I3)
M1 cards, each containing successive rows of array FSPEC (M1,3)	F3.0, 2(F12.9)
Array DSPX (M2)	26(I3)
Array DSPY (M22)	26(I3)
No cards, each containing successive rows of array H (NO,3)	3(E10.5)

OUTPUT

After an elastic solution is obtained, elastic stresses and strains for each element are printed in units of $\sigma_a/1000$ and $\sigma_a/1000E$, respectively, where σ_a is the applied uniaxial stress and E is Young's modulus.

For elastic-plastic behavior, accumulated stresses and strains are printed every tenth cycle. Stresses and strains are expressed in units of $S_y/1000$ and $S_y/1000E$, respectively, where S_y represents the applied stress necessary to initiate yielding. The quantity $S_y/1000$ equals YBAR/SBAR, where YBAR is the uniaxial yield stress and SBAR represents the largest effective stress obtained from the elastic solution. The accumulated applied stress, in units of S_y, and the element yielded, JYD, are printed after each cycle.

PROGRAM LISTING

The listing of the program and the subroutines follow after the next section.

TYPICAL INPUT DATA

The input data for the specimen analyzed follows the listing of the computer program subroutines.

```
      PROGRAM EPLAS5
C     ELASTO-PLASTIC SOLN FOR PLANE STRESS
      TYPE INTEGER BDW,DSPX,DSPY,S,X,Y,Z,EP,CYC,CYCMAX,S1,X1,X2,X3
      TYPE REAL INCR,INCR1
      DIMENSION CORD(227,2),NP(395,3),AREA(395),GE(3,3),DSPX(16),
     1B(3,6),DU(6),DE(3),DELM(395,5),SELM(395,5),DS(3),SELM1(395,5),
     2DELM1(395,5),HEFF(395),DSPY(16),FSPEC(11,3),THKNS(395),EP(395)
      COMMON /1/A(454,36),DELF(454),DELU(489)
      COMMON /A/H(30,3)
   67 FORMAT(///1X,3HCYC,2X,I4,10X,5HNITER,2X,I4)
   82 FORMAT(//1X,19HSTRAINS FOR EP CASE/4X,24HELEMENT--X,Y,XY,Z,EFF PL)
   83 FORMAT(1X,I4,4X,5(E16.9,2X))
   84 FORMAT(1H1,20HSTRESSES FOR EP CASE/4X,21HELEMENT--X,Y,XY,Z,EFF)
   90 FORMAT(1X,29HMAX DEVIATION FROM FLOW CURVE,2X,E16.9)
   88 FORMAT(///1X,12HNITER=NITMAX)
    1 FORMAT(F4.0,6(I4))
    2 FORMAT(2(I4),F5.0,2(E10.5),I4)
   47 FORMAT(6(2X,E16.9))
   11 FORMAT(//1X,6HCYCMAX,3X,6HNITMAX,3X,2HJ1,7X,4HSBAR,14X,4HYBAR,14X,
     11HE,14X,2HNO,3X,4HCRIT/(2X,I4,2X,I4,4X,I4,2X,E16.9,2X,E16.9,2X,
     2E16.9,3X,I4,2X,F7.4))
   13 FORMAT(E10.5,E10.5,E10.5)
  103 FORMAT(1X,27HTHIS ELEMENT IS NOW PLASTIC,2X,I4)
  108 FORMAT(12F6.4)
  111 FORMAT(26I3)
  113 FORMAT(F3.0,2F12.9)
  117 FORMAT(1H1,3X,2HPR,9X,1HM,5X,1HN/
     1(2X,F9.5,2X,I4,2X,I4///))
  118 FORMAT(1H1,16HELASTIC STRESSES/4X,17HELEMENT--X,Y,XY,Z)
  120 FORMAT(//1X,15HELASTIC STRAINS/4X,17HELEMENT--X,Y,XY,Z)
  139 FORMAT(1X,16HACCUMULATED LOAD,2X,F9.5)
C     ****************
C     READ INPUT DATA AND ZERO ARRAYS.
C     ****************
      READ(60,1) PR,BDW,N,M,M2,M22,M1
      READ(60,2) CYCMAX,NITMAX,CRIT,E,YBAR,NO
      N2=2*N
      M3=(BDW+1)/2
      M4=(BDW-1)/2
      DO 109 J=1,2
  109 READ(60,108) (CORD(I,J),I=1,N)
      DO 110 J=1,3
  110 READ(60,111) (NP(I,J),I=1,M)
      DO 112 I=1,M1
  112 READ(60,113) FSPEC(I,1),FSPEC(I,2),FSPEC(I,3)
      READ(60,111) (DSPX(I),I=1,M2)
      READ(60,111) (DSPY(I),I=1,M22)
      IF(CYCMAX.EQ.0) GO TO 173
      DO 172 I=1,NO
  172 READ(60,13) (H(I,J), J=1,3)
  173 CONTINUE
      DO 126 I=1,M
      HEFF(I)=THKNS(I)=0.0
      DELM(I,5)=0.0
  126 EP(I)=0
      DO 96 I=1,N
   96 DELU(2*I)=DELU(2*I-1)=DELF(2*I)=DELF(2*I-1)=0.0
      CYC=J1=0
      SBAR=0.0
```

```
      ACCUM=1.0
      NITER=1
      IF(CYC.EQ.0) GO TO 53
  138 CONTINUE
C     ***************
C     FROM THE ELASTIC SOLN. DETERMINE THAT ELEMENT(J1) HAVING THE
C     LARGEST EFFECTIVE STRESS(SBAR). ESTIMATE TANGENT MODULUS(HEFF)
C     FOR ELEMENT J1 FOR FIRST LOAD CYCLE.
C     ***************
      SBAR=0.0
      J1=0
      DO 51 I=1,M
      T1=(2.0*SELM(I,1)-SELM(I,2))/3.0
      T2=(2.0*SELM(I,2)-SELM(I,1))/3.0
      T3=(-SELM(I,1)-SELM(I,2))/3.0
      T4=SELM(I,3)
      SELM(I,5)=SQRTF(3.0*(T1**2+T2**2+T3**2+2.0*T4**2)/2.0)
      IF(SELM(I,5).LT.SBAR) GO TO 51
      SBAR=SELM(I,5)
      J1=I
   51 CONTINUE
      EP(J1)=1
      HEFF(J1)=H(1,1)
C     ***************
C     CALCULATE THICKNESS(THKNS) OF EACH ELEMENT AT INITIAL YIELD.
C     ***************
      DO 175 I=1,M
  175 THKNS(I)=1.0+DELM(I,4)*YBAR/(SBAR*E)
C     ***************
C     FORM OVERALL STIFFNESS MATRIX HAVING COMPONENTS A(I,J).
C     ***************
   53 CONTINUE
      DO 125 I=1,N2
      DO 125 J=1,M3
  125 A(I,J)=0.0
      DO 92 I=1,M
      X=NP(I,1)
      Y=NP(I,2)
      Z=NP(I,3)
      AREA(I)=0.5*(CORD(Z,2)*CORD(Y,1)-CORD(Z,2)*CORD(X,1)-CORD(X,2)*
     1            CORD(Y,1)+CORD(X,1)*CORD(Y,2)-CORD(Y,2)*CORD(Z,1)+
     2            CORD(Z,1)*CORD(X,2))
      S=1
C     FORM ELASTIC STRESS-STRAIN MATRIX IF APPROPRIATE.
      COEF=1.0/(1.0-PR**2)
      IF(EP(I).EQ.1) GO TO 56
      GE(1,1)=COEF
      GE(2,2)=GE(1,1)
      GE(1,2)=COEF*PR
      GE(2,1)=GE(1,2)
      GE(3,3)=1.0/(2.0*(1.0+PR))
      GE(1,3)=GE(3,1)=GE(3,2)=GE(2,3)=0.0
      GO TO 57
   56 CONTINUE
C     FORM ELASTIC-PLASTIC STRESS-STRAIN MATRIX IF APPROPRIATE.
      T1=(2.0*SELM(I,1)-SELM(I,2))/3.0
      T2=(2.0*SELM(I,2)-SELM(I,1))/3.0
      T4=SELM(I,3)
      F=9.0*E/(4.0*SELM(I,5)**2*HEFF(I))
```

```
      R=1.0+F*((T1+T2)**2) *COEF+2.0*F*(T4**2-T1*T2)/(1.0+PR)
      GE(1,1)=(1.0+F*T2**2+2.0*F*T4**2/(1.0+PR))*COEF/R
      GE(1,2)=(PR-F*T1*T2+2.0*PR*F*T4**2/(1.0+PR))*COEF/R
      GE(2,1)=GE(1,2)
      GE(1,3)=(-F*(PR*T2+T1)*T4/(1.0+PR))*COEF/R
      GE(3,1)=GE(1,3)
      GE(2,2)=(1.0+F*T1**2+2.0*F*T4**2/(1.0+PR))*COEF/R
      GE(2,3)=(-F*(PR*T1+T2)*T4/(1.0+PR))*COEF/R
      GE(3,2)=GE(2,3)
      GE(3,3)=(1.0-PR+F*(T1**2+2.0*PR*T1*T2+T2**2)/(1.0+PR))*
     1COEF/(2.0*R)
   57 CONTINUE
      COEF1=1.0 /(4.0*AREA(I))
      IF(CYC.GT.0) COEF1=COEF1*THKNS(I)
   58 CONTINUE
C     BBX,BBY,BBZ,CCX,CCY,CCZ ARE NON-ZERO COMPONENTS OF THE STRAIN-
C     DISPLACEMENT MATRIX,B.
      BBX=CORD(Y,2)-CORD(Z,2)
      BBY=CORD(Z,2)-CORD(X,2)
      BBZ=CORD(X,2)-CORD(Y,2)
      CCX=CORD(Z,1)-CORD(Y,1)
      CCY=CORD(X,1)-CORD(Z,1)
      CCZ=CORD(Y,1)-CORD(X,1)
C     SINCE THE OVERALL STIFFNESS MATRIX IS SYMMETRIC, CALCULATE
C     ONLY THOSE COMPONENTS LYING ON AND ABOVE DIAGONAL. THE
C     DIAGONAL COMPONENT OF ROW I IS DENOTED A(I,1).
      A(2*X-1,1)=COEF1   *(BBX*(BBX*GE(1,1)+CCX*GE(3,1))+CCX*(BBX*
     1GE(3,1)+CCX*GE(3,3)))+A(2*X-1,1)
      A(2*X-1,2)=COEF1   *(BBX*(CCX*GE(2,1)+BBX*GE(3,1))+CCX*(CCX*
     1GE(3,2)+BBX*GE(3,3)))+A(2*X-1,2)
      A(2*X,1)   =COEF1   *(CCX*(CCX*GE(2,2)+BBX*GE(3,2))+BBX*(CCX*
     1GE(2,3)+BBX*GE(3,3)))+A(2*X,1)
      IF (Y.LT.X) GO TO 59
      A(2*X-1,2*Y-2*X+1)=COEF1   *(BBX*(BBY*GE(1,1)+CCY*GE(3,1))+CCX*
     1(BBY*GE(3,1)+CCY*GE(3,3)))+A(2*X-1,2*Y-2*X+1)
      A(2*X,2*Y-2*X)     =COEF1   *(CCX*(BBY*GE(2,1)+CCY*GE(3,2))+BBX*
     1(BBY*GE(3,1)+CCY*GE(3,3)))+A(2*X,2*Y-2*X)
      A(2*X-1,2*Y-2*X+2)=COEF1   *(BBX*(CCY*GE(2,1)+BBY*GE(3,1))+CCX*
     1(CCY*GE(3,2)+BBY*GE(3,3)))+A(2*X-1,2*Y-2*X+2)
      A(2*X,2*Y-2*X+1)   =COEF1   *(CCX*(CCY*GE(2,2)+BBY*GE(3,2))+BBX*
     1(CCY*GE(3,2)+BBY*GE(3,3)))+A(2*X,2*Y-2*X+1)
   59 CONTINUE
      IF (Z.LT.X) GO TO 60
      A(2*X-1,2*Z-2*X+1)=COEF1   *(BBX*(BBZ*GE(1,1)+CCZ*GE(3,1))+CCX*
     1(BBZ*GE(3,1)+CCZ*GE(3,3)))+A(2*X-1,2*Z-2*X+1)
      A(2*X,2*Z-2*X)     =COEF1   *(CCX*(BBZ*GE(2,1)+CCZ*GE(3,2))+BBX*
     1(BBZ*GE(3,1)+CCZ*GE(3,3)))+A(2*X,2*Z-2*X)
      A(2*X-1,2*Z-2*X+2)=COEF1   *(BBX*(CCZ*GE(2,1)+BBZ*GE(3,1))+CCX*
     1(CCZ*GE(3,2)+BBZ*GE(3,3)))+A(2*X-1,2*Z-2*X+2)
      A(2*X,2*Z-2*X+1)   =COEF1   *(CCX*(CCZ*GE(2,2)+BBZ*GE(3,2))+BBX*
     1(CCZ*GE(3,2)+BBZ*GE(3,3)))+A(2*X,2*Z-2*X+1)
   60 CONTINUE
      IF(S.EQ.2) GO TO 62
      IF(S.EQ.3) GO TO 92
      X=NP(I,2)
      Y=NP(I,3)
      Z=NP(I,1)
      S=S+1
      GO TO 58
```

```
   62 CONTINUE
      X=NP(I,3)
      Y=NP(I,1)
      Z=NP(I,2)
      S=S+1
      GO TO 58
   92 CONTINUE
C     ****************
C     ZERO ROWS AND COLUMNS RELATING TO THOSE NODES HAVING SPECIFIED
C     DISPLACEMENTS EQUAL TO ZERO.
C     ****************
      DO 19 L=1,M2
      LL=DSPX(L)
      DELF(2*LL-1)=0.0
      DO 19 LL1=2,M3
      IF((2*LL-LL1).LE.0) GO TO 44
      A(2*LL-LL1,LL1)=0.0
   44 CONTINUE
      A(2*LL-1,LL1)=0.0
   19 CONTINUE
      DO 20 L=1,M22
      LL=DSPY(L)
      DELF(2*LL)=0.0
      DO 20 LL1=2,M3
      IF((2*LL-LL1+1).LE.0) GO TO 45
      A(2*LL-LL1+1,LL1)=0.0
   45 CONTINUE
      A(2*LL,LL1)=0.0
   20 CONTINUE
C     ****************
C     IMPOSE UNIT LOAD INCREMENT AT BOUNDARY NODES.
C     ****************
      DO 124 I=1,N
  124 DELU(2*I-1)=DELU(2*I)=DELF(2*I-1)=DELF(2*I)=0.0
      DO 18 I=1,M1
      M11=FSPEC(I,1)
      DELF(2*M11)=FSPEC(I,3)
   18 DELF(2*M11-1)=FSPEC(I,2)
      S1=1
C     ****************
C     USING GAUSSIAN ELIMINATION, SOLVE SYSTEM OF EQUATIONS FOR DISPLACEMENTS.
C     ****************
      CALL SOLVE (N2,BDW)
C     ****************
C     CALCULATE STRAIN AND STRESS INCREMENTS DUE TO UNIT LOAD.
C     ****************
      INCR=1.0
      DO 69 I4=1,M
      X=NP(I4,1)
      Y=NP(I4,2)
      Z=NP(I4,3)
C     FORM STRAIN-DISPLACEMENT MATRIX, B, AND CALCULATE STRAIN INCREMENTS.
      BBX=CORD(Y,2)-CORD(Z,2)
      BBY=CORD(Z,2)-CORD(X,2)
      BBZ=CORD(X,2)-CORD(Y,2)
      CCX=CORD(Z,1)-CORD(Y,1)
      CCY=CORD(X,1)-CORD(Z,1)
      CCZ=CORD(Y,1)-CORD(X,1)
      DO 70 I8=1,6
```

```
      DU(I8)=0.0
   70 CONTINUE
      DO 71 I5=1,3
      DO 71 J5=1,6
      B(I5,J5)=0.0
   71 CONTINUE
      B(3,2)=B(1,1)=BBX
      B(3,4)=B(1,3)=BBY
      B(3,6)=B(1,5)=BBZ
      B(3,1)=B(2,2)=CCX
      B(3,3)=B(2,4)=CCY
      B(3,5)=B(2,6)=CCZ
      DO 72 J6=1,3
      L1=NP(I4,J6)
      DU(2*J6-1)=DELU(2*L1-1)
   72 DU(2*J6)  =DELU(2*L1)
      DO 73 L2=1,3
      DE(L2)=0.0
      DO 74 K2=1,6
   74 DE(L2)=DE(L2)+((1.0/(2.0*AREA(I4)))*B(L2,K2)*DU(K2))
      DELM1(I4,L2)=DE(L2)
   73 DELM1(I4,5)=0.0
C     FORM APPROPRIATE CONSTITUTIVE MATRIX, GE, AND CALCULATE STRESS INCREMENTS.
      IF(EP(I4).EQ.1) GO TO 76
      GE(1,1)=COEF
      GE(2,2)=GE(1,1)
      GE(1,2)=COEF*PR
      GE(2,1)=GE(1,2)
      GE(3,3)=1.0/(2.0*(1.0+PR))
      GE(1,3)=GE(3,1)=GE(3,2)=GE(2,3)=0.0
      IF(CYC.EQ.0) GO TO 75
   76 CONTINUE
      T1=(2.0*SELM(I4,1)-SELM(I4,2))/3.0
      T2=(2.0*SELM(I4,2)-SELM(I4,1))/3.0
      T3=(-SELM(I4,1)-SELM(I4,2))/3.0
      T4=SELM(I4,3)
      IF(EP(I4).EQ.0) GO TO 75
      F=9.0*E/(4.0*SELM(I4,5)**2*HEFF(I4))
      R=1.0+F*((T1+T2)**2) *COEF+2.0*F*(T4**2-T1*T2)/(1.0+PR)
      GE(1,1)=(1.0+F*T2**2+2.0*F*T4**2/(1.0+PR))*COEF/R
      GE(1,2)=(PR-F*T1*T2+2.0*PR*F*T4**2/(1.0+PR))*COEF/R
      GE(2,1)=GE(1,2)
      GE(1,3)=(-F*(PR*T2+T1)*T4/(1.0+PR))*COEF/R
      GE(3,1)=GE(1,3)
      GE(2,2)=(1.0+F*T1**2+2.0*F*T4**2/(1.0+PR))*COEF/R
      GE(2,3)=(-F*(PR*T1+T2)*T4/(1.0+PR))*COEF/R
      GE(3,2)=GE(2,3)
      GE(3,3)=(1.0-PR+F*(T1**2+2.0*PR*T1*T2+T2**2)/(1.0+PR))*
     1COEF/(2.0*R)
   75 CONTINUE
      DO 77 L3=1,3
      DS(L3)=0.0
      DO 78 K3=1,3
   78 DS(L3)=DS(L3)+GE(L3,K3)*DE(K3)
   77 SELM1(I4,L3)=DS(L3)
      SELM(I4,4)=SELM1(I4,4)=0.0
      DELM1(I4,4)=EP(I4)*(F*T3*(T1*SELM1(I4,1)+T2*SELM1(I4,2)+
     12.0*T4*SELM1(I4,3)))-PR*(SELM1(I4,1)+SELM1(I4,2))
      IF(CYC.EQ.0) GO TO 99
```

```
      IF(EP(I4).EQ.1) GO TO 69
C     ****************
C     DETERMINE THE NEXT ELEMENT TO YIELD(JYD) AND THE LOAD INCREMENT
C     REQUIRED(INCR).
C     ****************
      T11=(2.0*SELM1(I4,1)-SELM1(I4,2))/3.0
      T22=(2.0*SELM1(I4,2)-SELM1(I4,1))/3.0
      T33=(-SELM1(I4,1)-SELM1(I4,2))/3.0
      T44=SELM1(I4,3)
      A=T1*T11+T2*T22+T3*T33+2.0*T4*T44
      B=T11**2+T22**2+T33**2+2.0*T44**2
      C=T1**2+T2**2+T3**2+2.0*T4**2
      INCR1=(-A+SQRTF(A**2-B*(C-2.0*SBAR**2/3.0)))/B
      IF(INCR1.GT.INCR) GO TO 99
      INCR=INCR1
      JYD=I4
   99 CONTINUE
   69 CONTINUE
      IF(CYC.EQ.0) GO TO 91
C     ****************
C     CALCULATE STRESS AND STRAIN CHANGE IN EACH ELEMENT DUE TO
C     LOAD INCREMENT, INCR.
C     ****************
      CONV=0.0
      DO 100 I4=1,M
      DO 101 J=1,4
      DELM1(I4,J)=DELM1(I4,J)*INCR
  101 SELM1(I4,J)=SELM1(I4,J)*INCR
      T1=(2.0*SELM(I4,1)-SELM(I4,2))/3.0
      T2=(2.0*SELM(I4,2)-SELM(I4,1))/3.0
      T3=(-SELM(I4,1)-SELM(I4,2))/3.0
      T4=SELM(I4,3)
      T11=(2.0*SELM1(I4,1)-SELM1(I4,2))/3.0
      T22=(2.0*SELM1(I4,2)-SELM1(I4,1))/3.0
      T33=(-SELM1(I4,1)-SELM1(I4,2))/3.0
      T44=SELM1(I4,3)
      SM=SQRTF(3.0*((T1+T11)**2+(T2+T22)**2+(T3+T33)**2+2.0*(T4+T44)**2)
     1/2.0)
      SELM1(I4,5)=SM-SELM(I4,5)
      IF(EP(I4).EQ.0) GO TO 100
C     ****************
C     DETERMINE DEGREE OF DEVIATION(CONV) FROM FLOW CURVE AND ADJUST
C     TANGENT MODULUS IF DEVIATION EXCEEDS PREDETERMINED AMOUNT(CRIT).
C     ****************
      E1=DELM1(I4,1)-(SELM1(I4,1)-PR*SELM1(I4,2))
      E2=DELM1(I4,2)-(SELM1(I4,2)-PR*SELM1(I4,1))
      E3=DELM1(I4,4)+PR*(SELM1(I4,1)+SELM1(I4,2))
      E4=DELM1(I4,3)-(1.0+PR)*SELM1(I4,3)*2.0
      EM=SQRTF(2.0*(E1**2+E2**2+E3**2+E4**2/2.0)/3.0)
      DELM1(I4,5)=EM
      EM=EM+DELM(I4,5)
      CALL SUBSEM(EM,SBAR,YBAR,NO,HEM,SEM,E)
      CONV1=ABSF((SM-SEM)/SEM)
      IF(CONV1.LT.CONV) GO TO 102
      CONV=CONV1
  102 CONTINUE
      IF(CONV1.LT.CRIT) GO TO 100
      HEFF(I4)=(SEM-SELM(I4,5))/DELM1(I4,5)*E
      S1=0
```

```
  100 CONTINUE
      IF(S1.EQ.1) GO TO 80
      IF(NITER.EQ.NITMAX) GO TO 94
      NITER=NITER+1
      GO TO 53
   80 CONTINUE
C     ****************
C     ADD STRESS AND STRAIN INCREMENTS TO INITIAL VALUES, ESTIMATE PLASTIC
C     MODULI FOR NEXT LOAD INCREMENT, AND UPDATE COORDINATES AND
C     ELEMENTAL THICKNESSES.
C     ****************
      P1=YBAR/(SBAR*E)
      DO 81 I=1,M
      DO 82 J=1,5
      SELM(I,J)=SELM(I,J)+SELM1(I,J)
   82 DELM(I,J)=DELM(I,J)+DELM1(I,J)
      THKNS(I)=THKNS(I)*(1.0+DELM1(I,4)*P1)
      IF(EP(I).EQ.0) GO TO 81
      EM=DELM(I,5)
      CALL SUBSEM(EM,SBAR,YBAR,NO,HEM,SEM,E)
      HEFF(I)=HEM
   81 CONTINUE
      EP(JYD)=1
      HEFF(JYD)=H(1,1)
      DO 132 I=1,N
      CORD(I,1)=CORD(I,1)+(DELU(2*I-1)*YBAR/(E*SBAR))*INCR
  132 CORD(I,2)=CORD(I,2)+(DELU(2*I)*YBAR/(E*SBAR))*INCR
C     ****************
C     PRINT OUT RESULTS FOR CYCLE COMPLETED. CYCLE COMPLETED(CYC), NO.
C     OF ITERATIONS REQUIRED(NITER), ELEMENT YIELDED, ACCUMULATED LOAD,
C     AND MAX DEVIATION FROM FLOW CURVE(CONV). ACCUMULATED STRESSES AND
C     STRAINS ARE PRINTED EVERY 10TH CYCLE.
C     ****************
      WRITE(61,67) CYC,NITER
      WRITE(61,103) JYD
      ACCUM=ACCUM+INCR
      WRITE(61,139) ACCUM
      WRITE(61,90) CONV
      I5=CYC/10
      F1=CYC/10.0
      F2=I5-F1
      IF(CYC.GT.0.AND.F2.NE.0.0) GO TO 171
      WRITE(61,82)
      DO 95 I=1,M
   95 WRITE(61,83) I,(DELM(I,J),J=1,5)
      WRITE(61,84)
      DO 93 I=1,M
   93 WRITE(61,83) I,(SELM(I,J),J=1,5)
  171 CONTINUE
      IF(CYC.EQ.CYCMAX) GO TO 127
      CYC=CYC+1
      IF(CYC.EQ.(CYCMAX+1)) GO TO 127
      NITER=1
      GO TO 53
   94 WRITE(61,88)
      WRITE(61,90) CONV
      WRITE(61,67) CYC,NITER
      GO TO 127
   91 CONTINUE
```

```
C      ****************
C      PRINT OUT RESULTS FOR ELASTIC SOLUTION. POISSONS RATIO(PR), NO. OF
C      ELEMENTS(M), NO. OF NODES(N), STRESSES AND STRAINS FOR ALL ELEMENTS.
C      ****************
       WRITE(61,117) PR,M,N
       WRITE(61,120)
       DO 119 I=1,M
       DO 144 J=1,4
   144 DELM(I,J)=DELM1(I,J)
   119 WRITE(61,83) I,(DELM1(I,J),J=1,4)
       WRITE(61,118)
       DO 121 I=1,M
       DO 143 J=1,4
   143 SELM(I,J)=SELM1(I,J)
   121 WRITE(61,83) I,(SELM1(I,J),J=1,4)
       CYC=Q=1
       IF(CYCMAX.GT.0) GO TO 138
       GO TO 129
C      ****************
C      PRINT OUT, MAX NO. OF CYCLES ALLOWED(CYCMAX), MAX NO. OF ITERATIONS
C      ALLOWED(NITMAX), FIRST ELEMENT TO YIELD(J1), MAX EFFECTIVE STRESS
C      FROM ELASTIC SOLN(SBAR), PROPORTIONAL LIMIT(YBAR), NO. OF POINTS
C      ENTERED ON FLOW CURVE(NO), MAX ALLOWABLE DEVIATION FROM FLOW CURVE(CRIT).
C      ****************
   127 WRITE(61,11) CYCMAX,NITMAX,J1,SBAR,YBAR,E,NO,CRIT
   129 CONTINUE
       END
```

8

```
      SUBROUTINE SUBSEM(EM1,SBAR1,YBAR1,NO1,HEM1,SEM1,E1)
      COMMON/A/H(30,3)
      EM11=EM1*YBAR1/(SBAR1*E1)
      IF(EM11.LT.H(NO1,3)) GO TO 3
      SEM11=(EM11-H(NO1,3))*H(NO1,1)+H(NO1,2)
      HEM1=H(NO1,1)
      GO TO 2
    3 DO 1 I=1,NO1
      IF(EM11.GE.H(I+1,3)) GO TO 1
      SEM11=(EM11-H(I,3))*H(I,1)+H(I,2)
      HEM1=H(I,1)
      GO TO 2
    1 CONTINUE
    2 SEM1=SEM11*SBAR1/YBAR1
      RETURN
      END
```

```
      SUBROUTINE SOLVE(N21,BDW1)
      TYPE INTEGER BDW1
      COMMON/1/A1(454,36),DELF1(454),DELU1(489)
      DIMENSION B1(36,36),C1(36)
      NL=N21-1
      M5=(BDW1+1)/2
      M7=M1=M5-1
      DO 25 I=1,M5
      DO 25 J=I,M5
   25 B1(I,J)=B1(J,I)=A1(I,J+1-I)
      DO 70 N=1,NL
      DO 5 I=1,M5
    5 A1(N,I)=C1(I)=B1(1,I)
      DO 20 I=1,M1
      R=-B1(I+1,1)/C1(1)
      DO 30 J=1,M1
   30 B1(I,J)=B1(I+1,J+1)+R*C1(J+1)
   20 DELF1(N+I)=DELF1(N+I)+R*DELF1(N)
      IF((N+M1).GE.N21) GO TO 50
      DO 60 I=1,M5
   60 B1(M5,I)=B1(I,M5)=A1(N+I,M5+1-I)
      GO TO 70
   50 M5=M1
      M1=M1-1
   70 CONTINUE
      DELU1(N21)=DELF1(N21)/B1(1,1)
      DO 6 I=2,N21
      M4=N21-I+1
      SUM=0.0
      DO 7 K=1,M7
    7 SUM=SUM+DELU1(M4+K)*A1(M4,K+1)
    6 DELU1(M4)=(DELF1(M4)-SUM)/A1(M4,1)
      RETURN
      END
```

```
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2240.0          1500.0
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2270.0          750.0
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5.5200E+025.5120E+017.64   E-03
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3.7900E+025.8990E+0115.79  E-03
3.6000E+026.1200E+0121.62  E-03
3.1900E+026.3290E+0127.43  E-03
2.7100E+026.5140E+0133.23  E-03
2.6300E+026.6710E+0139.02  E-03
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